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The realization of non-transitive Novikov algebras

Chengming Bai[1,2] **and Daoji Meng**[3]

[1] Theoretical Physics Division, Nankai Institute of Mathematics, Tianjin 300071, People's Republic of China
[2] Liu Hui Center for Applied Mathematics, Tianjin 300071, People's Republic of China
[3] Department of Mathematics, Nankai University, Tianjin 300071, People's Republic of China

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Abstract

Novikov algebras were introduced in connection with hydrodynamic-type Poisson brackets and Hamiltonian operators in the formal variational calculus. We have given a kind of realization of transitive Novikov algebras through the Novikov algebras given by S Gelfand and their compatible infinitesimal deformations in Bai and Meng (2001 *J. Phys. A: Math. Gen.* **34** 3363–72). As a further and continuous study, we extend this realization theory to the non-transitive Novikov algebras in the paper. In two and three dimensions, we find that all non-transitive Novikov algebras also can be realized as the Novikov algebras given by S Gelfand and their compatible infinitesimal deformations. Moreover, they have simpler formulae.

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1. Introduction

Hamiltonian operators have a close relation with certain algebraic structures [1–8]. Gel'fand and Diki [1, 2] introduced formal variational calculus and found certain interesting Poisson structures when they studied Hamiltonian systems related to certain nonlinear partial differential equations, such as KdV equations. In [3], more connections between Hamiltonian operators and certain algebraic structures were found. Dubrovin, Balanskii and Novikov [4–6] studied similar Poisson structures from another point of view. One of the algebraic structures appearing in [3] and [6], which was called 'Novikov algebra' by Osborn [9–14], was introduced in connection with hydrodynamic-type Poisson brackets as follows:

$$\{u^i(x),\ u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_{k=1}^{N} u_x^k b_k^{ij}(u(x))\delta(x-y). \tag{1.1}$$

A Novikov algebra A is a vector space over a field $\boldsymbol{F}$ with a bilinear product $(x,\ y) \to xy$ satisfying

$$(x_1,\ x_2,\ x_3) = (x_2,\ x_1,\ x_3) \tag{1.2}$$

and

$$(x_1x_2)x_3 = (x_1x_3)x_2 \tag{1.3}$$

for $x_1,\ x_2,\ x_3 \in A$, where

$$(x_1,\ x_2,\ x_3) = (x_1x_2)x_3 - x_1(x_2x_3). \tag{1.4}$$

Novikov algebras are a special class of left-symmetric algebras which only satisfy (1.2). Left-symmetric algebras are non-associative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones [15–18].

The commutator of a Novikov algebra (or a left-symmetric algebra) A

$$[x,\ y] = xy - yx \tag{1.5}$$

defines a (sub-adjacent) Lie algebra $\mathcal{G} = \mathcal{G}(A)$. Let L_x, R_x denote left and right multiplication respectively, i.e., $\mathrm{L}_x(y) = xy$, $\mathrm{R}_x(y) = yx$, $\forall x,\ y \in A$. Then for a Novikov algebra, the left multiplication operators form a Lie algebra and the right multiplication operators are commutative.

Zel'manov [19] gave a fundamental structure theory of a finite-dimensional Novikov algebra over an algebraically closed field with characteristic 0: a Novikov algebra A is called right-nilpotent or transitive if every R_x is nilpotent. Then by (1.3), a finite-dimensional Novikov algebra A contains a (unique) largest transitive ideal $N(A)$ (is called the radical of A) and the quotient algebra $A/N(A)$ is a direct sum of fields. The transitivity corresponds to the completeness of the affine manifolds in geometry [15, 16].

However, for such a non-associative algebra system, it is obviously difficult to give a more detailed structure theory, and needless to say a complete classification. This can be seen from the complicated classification of Novikov algebras in low dimensions [20]. As we said in [21], one of the reasons is that there is not a 'suitable' representation theory for Novikov algebras because they are not associative in general (hence their representations should have bimodule structures). Moreover, the classification of non-transitive Novikov algebras are still unknown even when we know their transitive radicals $N(A)$ since the extensions by $N(A)$ is essential in general [20]. So it is important to find some realizations of both transitive and non-transitive Novikov algebras at first, which will be useful to construct a general theory.

The first important kind of Novikov algebras was found as follows: let $(A, \cdot)$ be a commutative-associative algebra, and D be its derivation. Then the new product

$$a *_x b = a \cdot Db + x \cdot a \cdot b \tag{1.6}$$

makes $(A,\ *_x)$ become a Novikov algebra for $x = 0$ as per S Gelfand [3], for $x \in \boldsymbol{F}$ by Filipov [22] and for a fixed element $x \in A$ by Xu [13]. In [21], we show that the algebra $(A,\ *) = (A,\ *_0)$ given by S Gelfand is transitive. We also construct a deformation theory and the Novikov algebras given by Filipov and Xu are the special compatible infinitesimal deformations of ones given by S Gelfand. Moreover, we prove that the transitive Novikov algebras in dimensions of $\leqslant 3$ can be realized as the algebras defined by S Gelfand and their compatible infinitesimal deformations. We conjecture that this conclusion can be extended to higher dimensions.

Thus, as a further and continuous study, we will extend this realization theory to non-transitive Novikov algebras in this paper. Although the ideas and methods that we use in this paper are similar to those in [21], we would like to point out that these results are non-trivial, and together with transitive Novikov algebras can be used to construct a complete realization theory. Moreover, on comparison with the realizations of transitive Novikov algebras, we can see that the realizations of non-transitive Novikov algebras in low dimensions have simpler formulae, which will be very useful in any applications. The paper is organized as follows. In section 2 we briefly introduce the deformation theory of Novikov algebras and the realization of transitive Novikov algebras given in [21]. In section 3, we give the two-dimensional realization of non-transitive Novikov algebras. In section 4, we give the realization of three-dimensional non-transitive Novikov algebras. In section 5, we give some conclusions and conjectures based on the discussions in the previous sections.

2. The deformation of Novikov algebras and the realization of transitive Novikov algebras

For completeness, we briefly introduce the main results in [21] in this section. Let $(A, *)$ be a Novikov algebra, and $g_p : A \times A \to A$ be a bilinear product defined by

$$g_q(a, b) = a * b + qG_1(a, b) + q^2 G_2(a, b) + q^3 G_3(a, b) + \cdots \tag{2.1}$$

where G_i are bilinear products with $G_0(a, b) = a * b$. We call (A_q, g_q) a deformation of $(A, *)$ if (A_q, g_q) is a family of Novikov algebras. In particular, we call G_1 an infinitesimal deformation if the deformation is given by

$$g_q(a, b) = a * b + qG_1(a, b) \tag{2.2}$$

that is, $G_2 = G_3 = \cdots = 0$. G_1 is an infinitesimal deformation if and only if

$$\begin{aligned} G_1(a,\ b * c) - G_1(a * b,\ c) + G_1(b * a, c) \\ -G_1(b,\ a * c) + a * G_1(b, c) - G_1(a,\ b) * c \\ +G_1(b, a) * c - b * G_1(a, c) = 0 \end{aligned} \tag{2.3}$$

$$G_1(a, b) * c - G_1(a, c) * b + G_1(a * b,\ c) - G_1(a * c,\ b) = 0. \tag{2.4}$$

Moreover, G_1 is called a compatible infinitesimal deformation if G_1 is commutative. Any Novikov algebra and its compatible infinitesimal deformation have the same sub-adjacent Lie algebra. An infinitesimal deformation is called special if the family of Novikov algebras (A_q, g_q) defined by (2.2) is mutually isomorphic for $q \neq 0$.

As in the introduction, let $(A, \cdot)$ be a finite-dimensional commutative-associative algebra and D be its derivation. Then $(A, *)$ is a Novikov algebra with the product

$$a * b = a \cdot Db. \tag{2.5}$$

Moreover, $(A, *)$ is transitive. For a Novikov algebra $(A, *)$ defined by the above equation, both $G_1(a, b) = a \cdot b$ and $G_1(a, b) = x \cdot a \cdot b$ for a fixed $x \in A$ satisfy (2.3) and (2.4). So $(A, *_x)$ is a Novikov algebra with the product

$$a *_x b = a \cdot Db + x \cdot a \cdot b \tag{2.6}$$

for any $x \in \boldsymbol{F}$ and $x \in A$, and the Novikov algebras defined by (2.6) are the compatible infinitesimal deformations of the Novikov algebras defined by (2.5).

We have proved that the two- and three-dimensional transitive Novikov algebras can be realized as the Novikov algebras defined by (2.5) and (2.6), except (A6) with $l = 0$ and (A8) and (A10). For these exceptional cases, we have proved that they can be realized as special compatible infinitesimal deformations of some transitive Novikov algebras defined by (2.5).

3. The realization of non-transitive Novikov algebras in two dimensions

It is easy to see that the Novikov algebra $(A, *_x)$ defined by (2.6) is transitive if $(A, \cdot)$ is a nilpotent commutative-associative algebra. Thus, this means that we should suppose $(A, \cdot)$ is a non-nilpotent commutative-associative algebra if we try to find the non-transitive algebras through (2.6). In two dimensions, any non-transitive Novikov algebra over the complex-number field can be realized as a Novikov algebra defined by (2.6), which can be seen from the following table: recall that the (form) characteristic matrix of a Novikov algebra is defined as

$$\mathcal{A} = \begin{pmatrix} \sum_{k=1}^{n} a_{11}^{k} e_k & \cdots & \sum_{k=1}^{n} a_{1n}^{k} e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^{n} a_{n1}^{k} e_k & \cdots & \sum_{k=1}^{n} a_{nn}^{k} e_k \end{pmatrix} \tag{3.1}$$

where $\{e_i\}$ is a basis of A and $e_i e_j = \sum_{k=1}^{n} a_{ij}^{k} e_k$. Moreover, under the same basis, any derivation D of A can be determined by a matrix; that is,

$$D = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \qquad D(e_i) = \sum_{j=1}^{n} a_{ij} e_j. \tag{3.2}$$

Characteristic matrix of $(A, *_x)$	Characteristic matrix of $(A, \cdot)$	Derivation D	x
(N1) $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$	(N1) $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$	$D = 0$	$x = e_1 + e_2$
(N2) $\begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}$	(N2) $\begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}$	$D = 0$	$x = e_2$
(N3) $\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$	(N3) $\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$	$D = 0$	$x = e_2$
(N4) $\begin{pmatrix} 0 & e_1 \\ 0 & e_2 \end{pmatrix}$	(N3) $\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$	$D = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$	$x = e_2$
(N5) $\begin{pmatrix} 0 & e_1 \\ 0 & e_1 + e_2 \end{pmatrix}$	(N3) $\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$	$D = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$	$x = e_1 + e_2$
(N6) $\begin{pmatrix} 0 & e_1 \\ le_1 & e_2 \end{pmatrix}$ $l \neq 0,\ 1$	(N3) $\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$	$D = \begin{pmatrix} l-1 & 0 \\ 0 & 0 \end{pmatrix}$	$x = e_2$

The above table is obtained as follows: for every two-dimensional non-nilpotent commutative-associative algebra with basis $\{e_1, e_2\}$ and its corresponding derivation algebra (the classification has already been given in [21]), we can obtain a series of Novikov algebras $(A, *_x)$ for derivation D with parameters and $x = \lambda e_1 + \mu e_2$ through (2.6) (of course many of them are isomorphic [21]). Comparing all of them (for all non-nilpotent commutative-associative algebras in two dimensions) with the classification of non-transitive Novikov algebras in two dimensions which is given in [20], we can find they indeed include all two-dimensional non-transitive Novikov algebras. Moreover, for the characteristic matrix (in the first column) of every two-dimensional non-transitive Novikov algebra $(A, *)$ given in [20], we can choose a (not necessarily unique) commutative-associative algebra $(A, \cdot)$ with the corresponding characteristic matrix (in the second column) and a fixed derivation D (in the third column) and a fixed element x (in the fourth column) such that $e_i * e_j = e_i \cdot De_j + x \cdot e_i \cdot e_j$, which is listed in the above table.

4. The realization of non-transitive Novikov algebras in three dimensions

Let A be a three-dimensional non-transitive Novikov algebra over the complex-number field. Then using the same method as in section 3 and comparing the corresponding results with the classification of three-dimensional non-transitive Novikov algebras given in [20], we can find that A can be realized as a Novikov algebra defined by (2.6) except A is type (E1). Similarly, we can obtain the following table:

Characteristic matrix of $(A, *_x)$	Characteristic matrix of $(A, \cdot)$	Derivation D	x
(B1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	(B1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$D = 0$	$x = e_2 + e_3$
(B2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	(B2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$D = 0$	$x = e_2 + e_3$
(B3) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	(B2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_2 + e_3$
(B4) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ 0 & 0 & e_1 + e_3 \end{pmatrix}$	(B2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_1 + e_2 + e_3$
(B5) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ le_1 & 0 & e_3 \end{pmatrix}$ $l \neq 0,\ 1$	(B2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} l-1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_2 + e_3$
(C1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	(C1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$D = 0$	$x = e_3$
(C2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	(C2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$D = 0$	$x = e_3$
	(B2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$D = 0$	$x = e_3$
(C3) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	(C2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
	(B2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
(C4) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 + e_3 \end{pmatrix}$	(C2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_1 + e_3$
	(B2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_1 + e_3$
(C5) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ le_1 & 0 & e_3 \end{pmatrix}$ $l \neq 0,\ 1$	(C2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} l-1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
	(B2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} l-1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
(C6) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & 0 & e_3 \end{pmatrix}$	(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
(C7) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & 0 & e_3 + e_2 \end{pmatrix}$	(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_2 + e_3$
(C8) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & 0 & e_3 \end{pmatrix}$	(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
(C9) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & 0 & e_3 \end{pmatrix}$ $l \neq 1,\ 0$	(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} l-1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$

Characteristic matrix of $(A, *_x)$	Characteristic matrix of $(A, \cdot)$	Derivation D	x
Continued			
(C10) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & 0 & e_3+e_2 \end{pmatrix}$ $l \neq 1$	(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} l-1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_2 + e_3$
(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = 0$	$x = e_3$
(C12) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & le_2 & e_3 \end{pmatrix}$ $l \neq 0,\ 1$	(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & l-1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
(C13) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & ke_2 & e_3 \end{pmatrix}$ $l,\ k \neq 1,\ 0$	(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} l-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
(C14) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_1+e_2 & e_3 \end{pmatrix}$	(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
(C15) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & e_1+le_2 & e_3 \end{pmatrix}$ $l \neq 1,\ 0$	(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} l-1 & 0 & 0 \\ 1 & l-1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
(C16) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & e_1 & e_3 \end{pmatrix}$	(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
(C17) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & e_1 & e_3+e_2 \end{pmatrix}$	(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_2 + e_3$
(C18) $\begin{pmatrix} 0 & 0 & e_1+e_2 \\ 0 & 0 & e_2 \\ 0 & -e_2 & e_3 \end{pmatrix}$	(D2) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
(C19) $\begin{pmatrix} 0 & 0 & e_1+e_2 \\ 0 & 0 & e_2 \\ 0 & -e_2 & e_3+e_1 \end{pmatrix}$	(D2) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_1 + e_3$
(D1) $\begin{pmatrix} e_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	(D1) $\begin{pmatrix} e_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$D = 0$	$x = e_3$
(D2) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	(D2) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = 0$	$x = e_3$
(D3) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1+e_2 & e_2 & e_3 \end{pmatrix}$	(D2) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
(D4) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ \frac{1}{2}e_1 & 0 & e_3 \end{pmatrix}$	(D2) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$
(D5) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ \frac{1}{2}e_1 & 0 & e_3+e_2 \end{pmatrix}$	(D2) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = 2e_2 + e_3$
(D6) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & (2l-1)e_2 & e_3 \end{pmatrix}$ $l \neq \frac{1}{2},\ 1$	(D2) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$D = \begin{pmatrix} l-1 & 0 & 0 \\ 0 & 2(l-1) & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$x = e_3$

According to the classification of three-dimensional non-transitive Novikov algebras, there is only type (E1): $\begin{pmatrix} 0 & 0 & 0 \\ -e_1 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$ which cannot be of the form $(A, *_x)$ defined by (2.6). It is a (unique) trivial extension by the two-dimensional non-commutative radical $N(A) = T(3)$, i.e., it is the direct sum of T(3) and the field. It is isomorphic to a special (compatible) infinitesimal deformation of type (A9) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & 0 \end{pmatrix}$ with $G_1 = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which is isomorphic to (C1). Recall that (A9) can be realized as $(A, *)$, where $(A, \cdot)$ is (B2) and $D = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with $a_{11} \neq 0$.

5. Conclusion and discussion

From the discussion in the previous sections, we have the following conclusion and conjecture.

(a) All two- and three-dimensional Novikov algebras can be realized as the Novikov algebras defined by Gel'fand and their compatible infinitesimal deformations. We thus make the following conjecture that this conclusion can be extended to higher dimensions:

Conjecture 1. *All Novikov algebras can be realized as the algebras defined by Gel'fand and their compatible infinitesimal deformations.*

(b) In particular, except for one type (the direct sum of the non-associative radical and the field), any two- and three-dimensional non-transitive Novikov algebra can be realized through (2.6). Moreover, we find that the radical of such a Novikov algebra is commutative. Thus, we have the following conjecture:

Conjecture 2. *Non-transitive Novikov algebras with commutative radicals can be realized as the algebras defined by (2.6).*

A more general conjecture is:

Conjecture 3. *Non-transitive Novikov algebras with associative radicals can be realized as the algebras defined by (2.6).*

(c) We would like to point out that type (C8) just corresponds to the Poisson brackets of one-dimensional hydrodynamics that was discussed in [6].

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